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# ON THE EDGE-COLOURING PROPERTY FOR THE HEREDITARY CLOSURE OF A COMPLETE UNIFORM HYPERGRAPH II

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# On the edge-colouring property for the hereditary closure of a complete uniform hypergraph II *)

by

R. Tijdeman

ABSTRACT

It is shown that the collection of subsets of cardinality at most h of a fixed set of cardinality n possesses a parallelism (1-factorization) if $n = kh-1$, h is odd, and $\frac{1}{2}(h+1) \le k \le h-4$, thereby completing results of A.E. Brouwer.

KEYWORDS & PHRASES: *parallelism, 1-factorization, edge-colouring property*

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*) This report will be submitted for publication elsewhere

## INTRODUCTION

This note is written as a sequel to A.E. BROUWER [1], where definitions and relevant literature can be found. The results given there together with what will be proved below yield a complete characterization of the cases in which the hereditary closure of the complete h-uniform hypergraph on n vertices ($\hat{K}_n^h$) admits a parallelism (has a 1-factorization).

MAIN THEOREM.

a) *If* $n \le 2h$, *then* $\hat{K}_n^h$ *has a 1-factorization if and only if* $\hat{K}_n^{n-h-1}$ *has.*

b) *If* $n = kh+\ell$ *with* $-1 \le \ell \le h-2$ *and* $k \ge 2$, *then* $\hat{K}_n^h$ *has a 1-factorization if and only if*

(i) $\ell = 0$ *and* $k \ge h-2$ *or* (ii) $\ell = -1$ *and* $k \ge \frac{1}{2}h-1$.

Indeed, a) is given by [1], Theorem 5; the "only if" part of b) is given by Theorems 1,2,3 and 4 and the "if" part of b) (i) is shown in Theorems 6 and 8. So we may assume that $n = kh-1$ and $k \ge \frac{1}{2}h-1$. The Theorems 7,9,12, 10 and 11 prove the 1-factorizability of $\hat{K}_n^h$ for h even, for $k \ge h-3$ and h odd and for $k = \frac{1}{2}(h-1)$. In order to complete the proof of the main theorem we have to construct a 1-factorization of $\hat{K}_n^h$ when $n = kh-1$, h is odd and $\frac{1}{2}(h+1) \le k \le h-4$. This is done below.

I thank Dr. A.E. Brouwer for his very substantial contribution to the presentation of this paper.

## THE CONSTRUCTION

Let $n = kh-1$ with h odd, $h \ge 5$ and $k \ge \frac{1}{2}(h+1)$. Hence $k \ge 3$. We use 1-factors of the following types.

| | |
|---|---|
| Type $\alpha$: | $1*(h-1)+(k-1)*h$ |
| $\beta$: | $\frac{1}{2}(h+1)*(h-2)+(k-\frac{1}{2}(h-1))*h$ |
| $\gamma$: | $\frac{1}{2}(h-1)*(h-2)+2*(h-1)+(k-\frac{1}{2}(h+1))*h$ |
| $\delta$: | $1*(h-3)+\frac{1}{2}(h-3)*(h-2)+1*(h-1)+(k-\frac{1}{2}(h-1))*h$ |
| $\varepsilon$: | $2*(h-3)+\frac{1}{2}(h-5)*(h-2)+(k-\frac{1}{2}(h-3))*h$ |
| $\eta_i$ for i even: | $1*i+\frac{1}{2}i*(h-2)+1*(h-1)+(k-\frac{1}{2}i-1)*h$ |
| $\eta_i$ for i odd : | $1*i+\frac{1}{2}(i+1)*(h-2)+(k-\frac{1}{2}(i+1))*h$ |

$(1 \le i \le h-4)$

(Here $\sum c_i{*}i$ denotes a partition of the n-set into $c_i$ sets of size i $(1 \le i \le h)$. Note that indeed $n = \sum c_i i$ and that all $c_i$ are integral and non-negative.)

By Baranyai's theorem we can find a 1-factorization of $\hat{K}_n^h$ using $N_j$ 1-factors of type $\sum c_i^{(j)}{*}i$ if and only if $\sum_j N_j c_i^{(j)} = \binom{n}{i}$ for $i = 1,\dots,h$. We find the frequencies $N_\alpha, N_\beta, \dots, N_{\eta_i}$ of the 1-factors by the following process: Let the variables A,B,C and D denote the number of h-sets, (h-1)-sets, (h-2)-sets and (h-3)-sets not yet used for some 1-factor, and let $A_s, B_s, C_s, D_s$ be the value of A,B,C,D respectively after step s $(s = 1,2,3,4)$.

<u>Step 1</u>. Take $\binom{n}{i}$ 1-factors of type $\eta_i$ $(1 \le i \le h-4)$. This exhausts the i-sets with $1 \le i \le h-4$. Furthermore,

$$A_1 = \binom{n}{h} - \sum_{\substack{i\le h-4\\ i\text{ odd}}} (k-\tfrac{1}{2}(i+1))\binom{n}{i} - \sum_{\substack{i\le h-5\\ i\text{ even}}} (k-\tfrac{1}{2}i-1)\binom{n}{i},$$

$$B_1 = \binom{n}{h-1} - \sum_{\substack{i\le h-5\\ i\text{ even}}} \binom{n}{i},$$

$$C_1 = \binom{n}{h-2} - \sum_{\substack{i\le h-4\\ i\text{ odd}}} \tfrac{1}{2}(i+1)\binom{n}{i} - \sum_{\substack{i\le h-5\\ i\text{ even}}} \tfrac{1}{2}i\binom{n}{i},$$

$$D_1 = \binom{n}{h-3}.$$

Note that

$$|B_1(k-1)-A_1| = \Big|\sum_{\substack{i\le h-4\\ i\text{ odd}}} (k-\tfrac{1}{2}(i+1))\binom{n}{i} - \sum_{\substack{i\le h-5\\ i\text{ even}}} \tfrac{1}{2}i\binom{n}{i}\Big|$$

$$< (k-\tfrac{1}{2}(h-3))\binom{n}{h-4} < \binom{n}{h-3},$$

$$\binom{n}{h} - A_1 \le (k-1)\sum_{i\le h-4}\binom{n}{i} \le (k-1)\binom{n}{h-3}\Big\{\frac{h-3}{n-h+4} + \Big(\frac{h-3}{n-h+4}\Big)^2 + \dots\Big\}$$

$$\le \frac{(h-3)(k-1)}{n-2h+7}\binom{n}{h-3} \le \binom{n}{h-3} \le \frac{1}{(k-1)^3}\binom{n}{h},$$

$$\binom{n}{h-1} - B_1 \le \sum_{i\le h-5}\binom{n}{i} < \frac{1}{k-1}\binom{n}{h-4} \le \frac{1}{(k-1)^4}\binom{n}{h-1},$$

$$\binom{n}{h-2} - C_1 \le \frac{h-3}{2}\sum_{i\le h-4}\binom{n}{i} < \frac{h-3}{2}\cdot\frac{1}{k-1}\binom{n}{h-3} < \frac{1}{k-1}\binom{n}{h-2},$$

so that we did not take more (h-i)-sets ($i = 0,1,2$) than is allowed. We continue taking 1-factors, each time decreasing A,B,C,D by the appropriate amount.

Step 2. While $D > 1$ take a 1-factor of type $\delta$ if $B(k-1)-A > \frac{1}{4}h^2$
and a 1-factor of type $\varepsilon$ otherwise.
If $D=1$, then take a 1-factor of type $\delta$.

After this step $D=0$; we used all (h-3)-sets. Taking a 1-factor of type $\delta$ decreases $B(k-1)-A$ by $\frac{1}{2}(h-3) \geq 1$, while taking a 1-factor of type $\varepsilon$ increases $B(k-1)-A$ by $k-\frac{1}{2}(h-3) \geq 2$. Since $|B_1(k-1)-A_1| < D_1$ it follows that

$$0 < B_2(k-1) - A_2 \leq \tfrac{1}{4}h^2 + k - \tfrac{1}{2}(h-3).$$

Since $N_\delta + 2N_\varepsilon = \binom{n}{h-3}$, we have

$$A_1 - A_2 \leq (k-\tfrac{1}{2}(h-3))\binom{n}{h-3} \leq \frac{1}{(k-1)^2}\binom{n}{h},$$

$$B_1 - B_2 = N_\delta \leq \binom{n}{h-3} \leq \frac{1}{(k-1)^2}\binom{n}{h-1},$$

$$C_1 - C_2 \leq \tfrac{1}{2}(h-3)\binom{n}{h-3},$$

and hence,

$$\binom{n}{h-2} - C_2 \leq \frac{h-3}{2(k-1)}\binom{n}{h-3} + \tfrac{1}{2}(h-3)\binom{n}{h-3} \leq \frac{h-3}{2(k-1)} \cdot \frac{k(h-2)}{n-h+3}\binom{n}{h-2}$$

$$\leq (1-\frac{1}{k-1}) .1. \binom{n}{h-2} < \binom{n}{h-2}$$

so that $A_2, B_2$ and $C_2$ are positive. By $\binom{n}{h-2} \geq \binom{n}{2} \geq \frac{1}{2}h^2(k-1)^2$, we further obtain

$$C_2 \geq \frac{1}{k-1}\binom{n}{h-2} \geq \tfrac{1}{2}(k-1)h^2 \geq h^2.$$

Step 3. While $C > \frac{1}{4}h^2$ take a 1-factor of type $\gamma$ if $B(k-1)-A > \frac{1}{4}h^2$
and a 1-factor of type $\beta$ otherwise.

Taking a 1-factor of type $\gamma$ decreases $B(k-1)-A$ by $k+\frac{1}{2}(h-3)$ while taking a 1-factor of type $\beta$ increases $B(k-1)-A$ by $k-\frac{1}{2}(h-1) \geq 1$. Hence,

$$\tfrac{1}{4}h^2 - k - \tfrac{1}{2}(h-3) \leq B_3(k-1) - A_3 \leq \tfrac{1}{4}h^2 + k - \tfrac{1}{2}(h-1).$$

We further note that $C_3 \geq \frac{1}{4}h^2 - \frac{1}{2}(h+1) > \frac{1}{4}(h-1)^2-1$. Since any number larger than $\frac{1}{4}(h-1)^2-1$ is a non-negative linear combination of $\frac{1}{2}(h-1)$ and $\frac{1}{2}(h+1)$, the execution of the following step is possible.

Step 4. Take 1-factors $\beta$ and $\gamma$ in such a way that C becomes zero. This step at most increases $B(k-1)-A$ by $\frac{1}{2}(h-1)(k-\frac{1}{2}(h-1))$ or decreases it by $\frac{1}{2}(h-1)(k+\frac{1}{2}(h-3))$. Hence,

$$\tfrac{1}{4}h^2 - \tfrac{1}{2}(h+1)(k+\tfrac{1}{2}(h-3)) \leq B_4(k-1)-A_4 \leq \tfrac{1}{4}h^2 + \tfrac{1}{2}(h+1)(k-\tfrac{1}{2}(h-1)).$$

In steps 3 and 4 we take at most $C_2/(\frac{1}{2}(h-1))$ 1-factors and each of them diminishes B with at most 2. Hence,

$$B_2 - B_4 \leq \frac{4C_2}{h-1} < \frac{4}{h-1}\binom{n}{h-2}$$

and therefore, by $h \geq 5$ and $k \geq 3$,

$$\binom{n}{h-1} - B_4 \leq \left(\binom{n}{h-1}-B_1\right) + (B_1-B_2) + (B_2-B_4)$$
$$< \left(\frac{1}{(k-1)^4} + \frac{1}{(k-1)^2} + \frac{4}{h-1}\cdot\frac{1}{k-1}\right)\binom{n}{h-1} < \binom{n}{h-1}.$$

Thus $B_4 > 0$. Since $\sum_{i=1}^{h} i\binom{n}{i} = n\sum_{i=1}^{h}\binom{n-1}{i-1}$ is divisible by n, each partition covers n points and all i-sets except for h-sets and (h-1)-sets are exhausted, we have $(h-1)B_4+hA_4 \equiv 0 \pmod{n}$. Hence,

$$h(B_4(k-1)-A_4) = nB_4-((h-1)B_4+hA_4) \equiv 0 \pmod{n}.$$

By $(h,n) = 1$ it follows that $B_4(k-1)-A_4 \equiv 0 \pmod{n}$. However,

$$B_4(k-1)-A_4 \leq \tfrac{1}{4}h^2 + \tfrac{1}{2}(h+1)(k-\tfrac{1}{2}(h-1)) = \tfrac{1}{2}k(h+1) < n$$

and

$$B_4(k-1)-A_4 \geq \tfrac{1}{4}h^2 - \tfrac{1}{2}(h+1)(k+\tfrac{1}{2}(h-3)) = -\tfrac{1}{2}(h+1)(k-1) > -n.$$

From $B_4(k-1)-A_4 \equiv 0 \pmod{n}$ and $|B_4(k-1)-A_4| < n$ we conclude that $B_4(k-1) = A_4$. By $B_4 > 0$ we see that $A_4 > 0$. It is now obvious that the following step finishes the construction.

Step 5. Take $B_4$ 1-factors of type $\alpha$.

REMARK. In fact we proved the more general result:

Let $\{h-3,h-2,h-1,h\} \subset H \subset \{1,\dots,h\}$, $h$ odd, $h \geq 5$ and $k \geq \frac{1}{2}(h+1)$, $n = kh-1$. Then $K_n^H$ is 1-factorizable.

REFERENCE

[1] BROUWER, A.E., *On the edge-colouring property for the hereditary closure of a complete uniform hypergraph*, Report ZW 95/77, Math. Centr., Amsterdam, 1977.